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B.M.S COLLEGE FOR WOMEN, AUTONOMOUS
BENGALURU -560004
SEMESTER END EXAMINATION – APRIL/ MAY 2023

M.Sc. Mathematics – III Semester

DIFFERENTIAL GEOMETRY

Course Code: MM303T **QP Code: 13003**
Duration: 3 Hours **Max. Marks: 70**

Instructions: 1) All questions carry equal marks.
2) Answer any five full questions.

1. (a) Define directional derivative of a differentiable real valued function f on E^3. If $v_p = (v_1, v_2, v_3)_p$ is a tangent vector to E^3 then prove that $v_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f\,(p)}{\partial x_i}$ and deduce that $U_i[f] = \frac{\partial f}{\partial x_i}$, $i = 1,2,3$.
(b) If $V = xU_1 + yU_3$ and $W = 2x^2U_2 - U_3$, then compute $W - xV$ and find its value at the point $p = (-1,0,2)$.
(c) For any tangent vector v_p to E^3 at p, and for any functions f, g and any real constants a, b, prove that (i) $v_p[af + bg] = a\, v_p[f] + b\, v_p[g]$
(ii) $v_p[fg] = v_p[f]\, g(p) + f(p)\, v_p[g]$ **(5+3+6)**

2. (a) Explain reparameterization of a curve in E^3. If β is a reparameterization of a curve α by h, then prove that $\beta'(s) = \alpha'(h(s))\frac{dh(s)}{ds}$. Further verify the above formula for $\alpha(t) = (2\,cos^2 t, sin\,2t, 2\,sint)$ and $h(s) = sin^{-1}s$, $0 < s < 1$.
(b) Let f and g be functions, ϕ and ψ are 1-forms. Then prove that
$$d(\phi \wedge \psi) = (d\phi \wedge \psi) - (\phi \wedge d\psi)$$
Further verify the above formula for $\phi = \frac{dx}{y}$ and $\psi = z\,dy$. **(7+7)**

3. (a) Compute the Frenet apparatus K, τ, T, N, B of the unit speed curve
$$\beta(s) = \left(\frac{4}{5}\,cos\,s, 1 - sin\,s, \frac{-3}{5}\,cos\,s\right)$$
(b) Show that a curve lying on sphere of radius $'a'$ has curvature $k \geq \frac{1}{a}$.
(c) If $W = \sum W_i U_i$ and V is a vector field on E^3, then show that $\nabla_V W = \sum V[W_i]U_i$. Use it to compute $\nabla_V W$ for $V = (y - x)U_1 + xy\,U_3$ and $W = x^2 U_1 + yz\,U_3$. **(5+5+4)**

4. (a) Define derivative map F_* of the mapping $F: E^n \to E^m$. If $\beta = F(\alpha)$ is an image of a curve α in E^n, then prove that $\beta' = F_*(\alpha')$ and further show that F_* is a linear transformation.

(b) If $A = (a_{ij}), i,j = 1,2,3$ and $W = (w_{ij})$ are respectively attitude matrix and matrix of connection forms of a frame field E_1, E_2, E_3 then prove that $W = (dA)A^t$.
(c) Compute the connection forms for a cylindrical frame field. **(6+4+4)**

5. (a) Define proper patch. Verify that $X(u,v) = (u, uv, v)$ is a patch.
(b) Define parametrization of a region. Obtain parametrization of torus of revolution.
(c) If g is a real valued differentiable function on E^3 and C is a real constant, then prove that $M = \{(x,y,z) \in E^3 : g(x,y,z) = C\}$ is a surface in E^3 provided $dg \neq 0$ at any point of M. Use it to prove that sphere in E^3 is a surface in E^3. **(4+4+6)**

6. (a) If X is a patch in a surface M and X_* is a derivative map of X. Show that $X_*(U_1) = X_u$ and
$X_*(U_2) = X_v$, where U_1, U_2 is the natural frame field on E^3.
(b) Let $F: M \to N$ be a mapping of surfaces and let ξ and η be forms on N. Then prove the following (i) $F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$
(ii) $F^*(d\xi) = d(F^*\xi)$ **(7+7)**

7. (a) Let α be a curve in $M \subseteq E^3$. If U is a unit normal to M restricted to the curve α. Then show
that $S(\alpha') = -U'$ and $\alpha''.U = S(\alpha').\alpha'$
(b) Define an umbilic point. If p is an umbilic point of a surface M in E^3, then show that the shape operator S at p is just scalar multiplication by $k = k_1 = k_2$, where k_1 and k_2 are principal curvatures of M.
(c) Define Gaussian and mean curvature. Show that $k = k_1 k_2, H = \frac{1}{2}(k_1 + k_2)$. **(4+7+3)**

8. (a) With usual notations prove that $k(x) = \frac{ln - m^2}{EG - F^2}$, $H(x) = \frac{Gl + En - 2Fm}{2(EG - F^2)}$.
(b) Compute k, H, k_1, k_2 for the surface Helicoid $X(u,v) = (u\,cosv,\ u\,sinv,\ bv), b \neq 0$.
(c) Determine the geodesics in planes and spheres in E^3. **(4+6+4)**

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